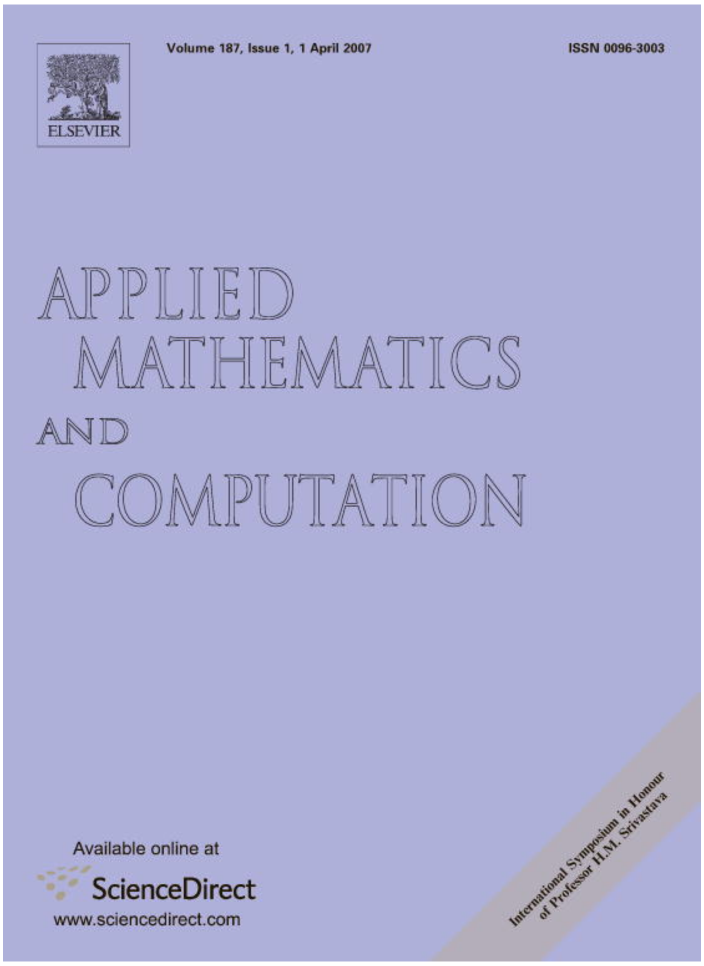
Volume 187, Issue 1, 1 April 2007
ISSN 0096-3003
ELSEVIER
APPLIED MATHEMATICS AND COMPUTATION
International Symposium in Honour of Professor H.M. Srivastava
Available online at
ScienceDirect
www.sciencedirect.com

Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 187 (2007) 35–46
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Coefficient bounds for p-valent functions

Rosihan M. Ali *, V. Ravichandran, N. Seenivasagan

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia

Dedicated to Professor H.M. Srivastava on the occasion of his 65th birthday

Abstract

Sharp bounds for $|a_{p+2} - \mu a_{p+1}^2|$ and $|a_{p+3}|$ are derived for certain p-valent analytic functions. These are applied to obtain Fekete-Szegö like inequalities for several classes of functions defined by convolution.

Keywords: Analytic functions; Starlike functions; Convex functions; p-Valent functions; Subordination; Convolution; Fekete-Szegö inequalities

1. Introduction

Let $\mathscr{A}_p$ be the class of all functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\Delta := \{z : |z| < 1\}$ and let $\mathscr{A} := \mathscr{A}_1$. For $f(z)$ given by (1) and $g(z)$ given by $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$, their convolution (or Hadamard product), denoted by $f * g$, is defined by

$$(f * g)(z) := z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

The function $f(z)$ is subordinate to the function $g(z)$, written $f(z) \prec g(z)$, provided there is an analytic function $w(z)$ defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Let φ be an analytic function with positive real part in the unit disk Δ with $\varphi(0) = 1$ and $\varphi'(0) > 0$ that maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. We define the class $S^*_{b,p}(\varphi)$ to be the subclass of $\mathscr{A}_p$ consisting of functions $f(z)$ satisfying

* Corresponding author.
E-mail addresses: rosihan@cs.usm.my (R.M. Ali), vravi@cs.usm.my (V. Ravichandran), vasagan2000@yahoo.co.in (N. Seenivasagan).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2006.08.100

$$1+\frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)}-1\right)\prec\varphi(z),\quad (z\in\Delta \text{ and } b\in\mathbb{C}\setminus\{0\}).$$

As special cases, let

$$S_p^*(\varphi):=S_{1,p}^*(\varphi),\qquad S_b^*(\varphi):=S_{b,1}^*(\varphi),\qquad S^*(\varphi):=S_{1,1}^*(\varphi).$$

For a fixed analytic function $g\in\mathscr{A}_p$ with positive coefficients, define the class $S_{b,p,g}^*(\varphi)$ to be the class of all functions $f\in\mathscr{A}_p$ satisfying $f*g\in S_{b,p}^*(\varphi)$. This class includes as special cases several other classes studied in the literature. For example, when $g(z)=z^p+\sum_{n=p+1}^{\infty}\frac{n}{p}z^n$, the class $S_{b,p,g}^*(\varphi)$ reduces to the class $C_{b,p}(\varphi)$ consisting of functions $f\in\mathscr{A}_p$ satisfying

$$1-\frac{1}{b}+\frac{1}{bp}\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\varphi(z),\quad (z\in\Delta \text{ and } b\in\mathbb{C}\setminus\{0\}).$$

The classes $S^*(\varphi)$ and $C(\varphi):=C_{1,1}(\varphi)$ were introduced and studied by Ma and Minda [9].

Define the class $R_{b,p}(\varphi)$ to be the class of all functions $f\in\mathscr{A}_p$ satisfying

$$1+\frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}}-1\right)\prec\varphi(z),\quad (z\in\Delta \text{ and } b\in\mathbb{C}\setminus\{0\}),$$

and for a fixed function g with positive coefficients, let $R_{b,p,g}(\varphi)$ be the class of all functions $f\in\mathscr{A}_p$ satisfying $f*g\in R_{b,p}(\varphi)$.

Several authors [4,8,10,16,17,12] have studied the classes of analytic functions defined by using the expression $\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}$. We shall also consider a class defined by the corresponding quantity for p-valent functions. Define the class $S_p^*(\alpha,\varphi)$ to be the class of all functions $f\in\mathscr{A}_p$ satisfying

$$\frac{1+\alpha(1-p)}{p}\frac{zf'(z)}{f(z)}+\frac{\alpha}{p}\frac{z^2f''(z)}{f(z)}\prec\varphi(z)\quad (z\in\Delta \text{ and } \alpha\geqslant 0).$$

Note that $S_p^*(0,\varphi)$ is the class $S_p^*(\varphi)$. Let $S_{p,g}^*(\alpha,\varphi)$ be the class of all functions $f\in\mathscr{A}_p$ for which $f*g\in S_p^*(\alpha,\varphi)$.

We shall also consider the class $L_p^M(\alpha,\varphi)$ consisting of p-valent α-convex functions with respect to φ. These are functions $f\in\mathscr{A}_p$ satisfying

$$\frac{1-\alpha}{p}\frac{zf'(z)}{f(z)}+\frac{\alpha}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\varphi(z)\quad (z\in\Delta \text{ and } \alpha\geqslant 0).$$

Further let $M_p(\alpha,\varphi)$ be the class of functions $f\in\mathscr{A}_p$ satisfying

$$\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\prec\varphi(z)\quad (z\in\Delta \text{ and } \alpha\geqslant 0).$$

Functions in this class are called logarithmic p-valent α-convex functions with respect to φ.

In this paper, we obtain Fekete-Szegö inequalities and bounds for the coefficient a_{p+3} for the classes $S_p^*(\varphi)$ and $S_{p,g}^*(\varphi)$. These results are then extended to the other classes defined earlier. See [1–7,9,13,14,18] for Fekete-Szegö problem for certain related classes of functions.

Let Ω be the class of analytic functions of the form

$$w(z)=w_1z+w_2z^2+\cdots \tag{2}$$

in the unit disk Δ satisfying the condition $|w(z)|<1$. We need the following lemmas to prove our main results.

Lemma 1. *If* $w\in\Omega$, *then*

$$|w_2-tw_1^2|\leqslant\begin{cases}-t & \text{if } t\leqslant -1,\\ 1 & \text{if } -1\leqslant t\leqslant 1,\\ t & \text{if } t\geqslant 1.\end{cases}$$

When $t<-1$ *or* $t>1$, *equality holds if and only if* $w(z)=z$ *or one of its rotations. If* $-1<t<1$, *then equality holds if and only if* $w(z)=z^2$ *or one of its rotations. Equality holds for* $t=-1$ *if and only if* $w(z)=$

$z\frac{\lambda+z}{1+\lambda z}$ $(0 \leqslant \lambda \leqslant 1)$ or one of its rotations while for $t = 1$, equality holds if and only if $w(z) = -z\frac{\lambda+z}{1+\lambda z}$ $(0 \leqslant \lambda \leqslant 1)$ or one of its rotations.

Also the sharp upper bound above can be improved as follows when $-1 < t < 1$:

$$|w_2 - tw_1^2| + (t+1)|w_1|^2 \leqslant 1 \quad (-1 < t \leqslant 0)$$

and

$$|w_2 - tw_1^2| + (1-t)|w_1|^2 \leqslant 1 \quad (0 < t < 1).$$

Lemma 1 is a reformulation of a Lemma of Ma and Minda [9].

Lemma 2 [5, Inequality 7, p. 10]. *If $w \in \Omega$, then, for any complex number t,*

$$|w_2 - tw_1^2| \leqslant \max\{1; |t|\}.$$

The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

Lemma 3 [11]. *If $w \in \Omega$, then for any real numbers q_1 and q_2, the following sharp estimate holds:*

$$|w_3 + q_1 w_1 w_2 + q_2 w_1^3| \leqslant H(q_1, q_2), \tag{3}$$

where

$$H(q_1,q_2) = \begin{cases} 1 & \text{for } (q_1,q_2) \in D_1 \cup D_2, \\ |q_2| & \text{for } (q_1,q_2) \in \bigcup_{k=3}^{7} D_k, \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_8 \cup D_9, \\ \frac{1}{3}q_2\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{10} \cup D_{11} - \{\pm 2, 1\}, \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{12}. \end{cases}$$

The extremal functions, up to rotations, are of the form

$$w(z) = z^3, \qquad w(z) = z, \qquad w(z) = w_0(z) = \frac{z([(1-\lambda)\varepsilon_2 + \lambda\varepsilon_1] - \varepsilon_1\varepsilon_2 z)}{1 - [(1-\lambda)\varepsilon_1 + \lambda\varepsilon_2]z},$$

$$w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z}, \qquad w(z) = w_2(z) = \frac{z(t_2 + z)}{1 + t_2 z},$$

$$|\varepsilon_1| = |\varepsilon_2| = 1, \qquad \varepsilon_1 = t_0 - e^{\frac{-i\theta_0}{2}}(a \mp b), \qquad \varepsilon_2 = -e^{\frac{-i\theta_0}{2}}(ia \pm b),$$

$$a = t_0 \cos\frac{\theta_0}{2}, \qquad b = \sqrt{1 - t_0^2 \sin^2\frac{\theta_0}{2}}, \qquad \lambda = \frac{b \pm a}{2b},$$

$$t_0 = \left[\frac{2q_2(q_1^2+2) - 3q_1^2}{3(q_2-1)(q_1^2-4q_2)}\right]^{\frac{1}{2}}, \qquad t_1 = \left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}},$$

$$t_2 = \left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}}, \qquad \cos\frac{\theta_0}{2} = \frac{q_1}{2}\left[\frac{q_2(q_1^2+8) - 2(q_1^2+2)}{2q_2(q_1^2+2) - 3q_1^2}\right].$$

The sets D_k, $k = 1, 2, \ldots, 12$, *are defined as follows*:

$$D_1 = \left\{(q_1, q_2) : |q_1| \leqslant \frac{1}{2}, |q_2| \leqslant 1\right\},$$

$$D_2 = \left\{(q_1, q_2) : \frac{1}{2} \leqslant |q_1| \leqslant 2, \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1) \leqslant q_2 \leqslant 1\right\},$$

$$D_3 = \left\{(q_1, q_2) : |q_1| \leqslant \frac{1}{2}, q_2 \leqslant -1\right\},$$

$$D_4 = \left\{(q_1, q_2) : |q_1| \geqslant \frac{1}{2}, q_2 \leqslant -\frac{2}{3}(|q_1|+1)\right\},$$

$$D_5 = \{(q_1, q_2) : |q_1| \leqslant 2, q_2 \geqslant 1\},$$

$$D_6 = \left\{(q_1, q_2) : 2 \leqslant |q_1| \leqslant 4, q_2 \geqslant \frac{1}{12}(q_1^2+8)\right\},$$

$$D_7 = \left\{(q_1, q_2) : |q_1| \geqslant 4, q_2 \geqslant \frac{2}{3}(|q_1|-1)\right\},$$

$$D_8 = \left\{(q_1, q_2) : \frac{1}{2} \leqslant |q_1| \leqslant 2, -\frac{2}{3}(|q_1|+1) \leqslant q_2 \leqslant \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1)\right\},$$

$$D_9 = \left\{(q_1, q_2) : |q_1| \geqslant 2, -\frac{2}{3}(|q_1|+1) \leqslant q_2 \leqslant \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\right\},$$

$$D_{10} = \left\{(q_1, q_2) : 2 \leqslant |q_1| \leqslant 4, \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \leqslant q_2 \leqslant \frac{1}{12}(q_1^2+8)\right\},$$

$$D_{11} = \left\{(q_1, q_2) : |q_1| \geqslant 4, \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \leqslant q_2 \leqslant \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\right\},$$

$$D_{12} = \left\{(q_1, q_2) : |q_1| \geqslant 4, \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4} \leqslant q_2 \leqslant \frac{2}{3}(|q_1|-1)\right\}.$$

2. Coefficient bounds

By making use of the Lemmas 1–3, we prove the following bounds for the class $S^*_{p,g}(\varphi)$:

Theorem 1. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, *and*

$$\sigma_1 := \frac{B_2 - B_1 + pB_1^2}{2pB_1^2}, \qquad \sigma_2 := \frac{B_2 + B_1 + pB_1^2}{2pB_1^2}, \qquad \sigma_3 := \frac{B_2 + pB_1^2}{2pB_1^2}.$$

If f(z) given by (1) *belongs to* $S^*_p(\varphi)$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant \begin{cases} \frac{p}{2}\left(B_2 + (1-2\mu)pB_1^2\right) & \text{if } \mu \leqslant \sigma_1, \\ \frac{pB_1}{2} & \text{if } \sigma_1 \leqslant \mu \leqslant \sigma_2, \\ -\frac{p}{2}\left(B_2 + (1-2\mu)pB_1^2\right) & \text{if } \mu \geqslant \sigma_2. \end{cases} \tag{4}$$

Further, if $\sigma_1 \leqslant \mu \leqslant \sigma_3$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{1}{2pB_1}\left(1 - \frac{B_2}{B_1} + (2\mu-1)pB_1\right)|a_{p+1}|^2 \leqslant \frac{pB_1}{2}. \tag{5}$$

If $\sigma_3 \leqslant \mu \leqslant \sigma_2$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{1}{2pB_1}\left(1 + \frac{B_2}{B_1} - (2\mu-1)pB_1\right)|a_{p+1}|^2 \leqslant \frac{pB_1}{2}. \tag{6}$$

For any complex number μ,

$$\left|a_{p+2}-\mu a_{p+1}^{2}\right| \leqslant \frac{pB_1}{2}\max\left\{1;\left|\frac{B_2}{B_1}+(1-2\mu)pB_1\right|\right\}. \tag{7}$$

Further,

$$|a_{p+3}| \leqslant \frac{pB_1}{3}H(q_1,q_2), \tag{8}$$

where $H(q_1,q_2)$ *is as defined in* Lemma 3,

$$q_1 := \frac{4B_2+3pB_1^2}{2B_1} \quad \textit{and} \quad q_2 := \frac{2B_3+3pB_1B_2+p^2B_1^3}{2B_1}.$$

These results are sharp.

Proof. If $f(z)\in S_p^*(\varphi)$, then there is an analytic function $w(z)=w_1z+w_2z^2+\cdots\in\Omega$ such that

$$\frac{zf'(z)}{pf(z)}=\varphi(w(z)). \tag{9}$$

Since

$$\frac{zf'(z)}{pf(z)}=1+\frac{a_{p+1}}{p}z+\left(-\frac{a_{p+1}^2}{p}+\frac{2a_{p+2}}{p}\right)z^2+\left(\frac{3a_{p+3}}{p}-\frac{3}{p}a_{p+1}a_{p+2}+\frac{a_{p+1}^3}{p}\right)z^3+\cdots,$$

we have from (9),

$$a_{p+1}=pB_1w_1, \tag{10}$$

$$a_{p+2}=\frac{1}{2}\{pB_1w_2+p(B_2+pB_1^2)w_1^2\} \tag{11}$$

and

$$a_{p+3}=\frac{pB_1}{3}\left\{w_3+\frac{4B_2+3pB_1^2}{2B_1}w_1w_2+\frac{2B_3+3pB_1B_2+p^2B_1^3}{2B_1}w_1^3\right\}. \tag{12}$$

Using (10) and (11), we have

$$a_{p+2}-\mu a_{p+1}^2=\frac{pB_1}{2}\{w_2-vw_1^2\}, \tag{13}$$

where

$$v := \left[pB_1(2\mu-1)-\frac{B_2}{B_1}\right].$$

The results (4)–(6) are established by an application of Lemma 1, inequality (7) by Lemma 2 and (8) follows from Lemma 3.

To show that the bounds in (4)–(6) are sharp, we define the functions $K_{\varphi n}$ $(n=2,3,\ldots)$ by

$$\frac{zK'_{\varphi n}(z)}{pK_{\varphi n}(z)}=\varphi(z^{n-1}), \quad K_{\varphi n}(0)=0=[K_{\varphi n}]'(0)-1$$

and the functions F_λ and G_λ $(0\leqslant\lambda\leqslant 1)$ by

$$\frac{zF'_\lambda(z)}{pF_\lambda(z)}=\varphi\left(\frac{z(z+\lambda)}{1+\lambda z}\right), \quad F_\lambda(0)=0=F'_\lambda(0)-1$$

and

$$\frac{zG'_\lambda(z)}{pG_\lambda(z)}=\varphi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right), \quad G_\lambda(0)=0=G'_\lambda(0)-1.$$

Clearly the functions $K_{\varphi n},F_\lambda,G_\lambda\in S_p^*(\varphi)$. We shall also write $K_\varphi := K_{\varphi 2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then equality holds if and only if f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then equality holds if and only if f is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$ then equality holds if and only if f is F_λ or one of its rotations. Equality holds for $\mu = \sigma_2$ if and only if f is G_λ or one of its rotations. □

Corollary 1. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, *and let*

$$\sigma_1 := \frac{g_{p+1}^2}{g_{p+2}} \frac{B_2 - B_1 + pB_1^2}{2pB_1^2}, \qquad \sigma_2 := \frac{g_{p+1}^2}{g_{p+2}} \frac{B_2 + B_1 + pB_1^2}{2pB_1^2}, \qquad \sigma_3 := \frac{g_{p+1}^2}{g_{p+2}} \frac{B_2 + pB_1^2}{2pB_1^2}.$$

If $f(z)$ *given by* (1) *belongs to* $S^*_{p,g}(\varphi)$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant \begin{cases} \frac{p}{2g_{p+2}}\left(B_2 + \left(1 - 2\frac{g_{p+2}}{g_{p+1}^2}\mu\right)B_1^2\right) & \text{if } \mu \leqslant \sigma_1, \\ \frac{pB_1}{2g_{p+2}} & \text{if } \sigma_1 \leqslant \mu \leqslant \sigma_2, \\ -\frac{p}{2g_{p+2}}\left(B_2 + \left(1 - 2\frac{g_{p+2}}{g_{p+1}^2}\mu\right)B_1^2\right) & \text{if } \mu \geqslant \sigma_2. \end{cases}$$

Further, if $\sigma_1 \leqslant \mu \leqslant \sigma_3$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{g_{p+1}^2}{2g_{p+2}pB_1}\left(1 - \frac{B_2}{B_1} + \left(2\frac{g_{p+2}}{g_{p+1}^2}\mu - 1\right)B_1\right)|a_{p+1}|^2 \leqslant \frac{pB_1}{2g_{p+2}}.$$

If $\sigma_3 \leqslant \mu \leqslant \sigma_2$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{g_{p+1}^2}{2g_{p+2}pB_1}\left(1 + \frac{B_2}{B_1} - \left(2\frac{g_{p+2}}{g_{p+1}^2}\mu - 1\right)B_1\right)|a_{p+1}|^2 \leqslant \frac{pB_1}{2g_{p+2}}.$$

For any complex number μ,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant \frac{pB_1}{2g_{p+2}} \max\left\{1; \left|\frac{B_2}{B_1} + \left(1 - 2\frac{g_{p+2}}{g_{p+1}^2}\mu\right)B_1\right|\right\}.$$

Further,

$$|a_{p+3}| \leqslant \frac{pB_1}{3g_{p+3}} H(q_1, q_2), \tag{14}$$

where $H(q_1, q_2)$ *is as defined in Lemma* 3,

$$q_1 := \frac{4B_2 + 3pB_1^2}{2B_1} \quad \text{and} \quad q_2 := \frac{2B_3 + 3pB_1B_2 + p^2B_1^3}{2B_1}.$$

These results are sharp.

Theorem 2. *Let* φ *be as in Theorem* 1. *If* $f(z)$ *given by* (1) *belongs to* $S^*_{b,p,g}(\varphi)$, *then, for any complex number* μ, *we have*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant \frac{p|b|B_1}{2g_{p+2}} \max\left\{1; \left|\frac{B_2}{B_1} + bp\left(1 - 2\frac{g_{p+1}^2}{g_{p+2}}\mu\right)B_1\right|\right\}.$$

The result is sharp.

Proof. The proof is similar to the proof of Theorem 1. □

Proceeding similarly, we now obtain coefficient bounds for the class $R_{b,p,g}(\varphi)$.

Theorem 3. *Let $\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$ If $f(z)$ given by (1) belongs to $R_{b,p}(\varphi)$, then for any complex number μ,*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant |\gamma| \max\{1; |v|\}, \tag{15}$$

where

$$v := \mu \frac{pbB_1(p+2)}{(p+1)^2} - \frac{B_2}{B_1} \quad \text{and} \quad \gamma := \frac{bpB_1}{2+p}.$$

Further,

$$|a_{p+3}| \leqslant \frac{|b|pB_1}{3+p} H(q_1, q_2), \tag{16}$$

where $H(q_1, q_2)$ is as defined in Lemma 3,

$$q_1 := \frac{2B_2}{B_1} \quad \text{and} \quad q_2 := \frac{B_3}{B_1}.$$

These results are sharp.

Proof. A computation shows that

$$1 + \frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}} - 1\right) = 1 + \frac{p+1}{bp} a_{p+1}z + \frac{p+2}{bp} a_{p+2}z^2 + \frac{p+3}{bp} a_{p+3}z^3 + \cdots.$$

Thus

$$a_{p+2} - \mu a_{p+1}^2 = \frac{bpB_1}{p+2}\{w_2 - vw_1^2\} = \gamma\{w_2 - vw_1^2\}, \tag{17}$$

where $v := \left[\frac{bpB_1\mu(2+p)}{(p+1)^2} - \frac{B_2}{B_1}\right]$ and $\gamma := \frac{bpB_1}{p+2}$. The result now follows from Lemmas 2 and 3. □

Remark 1. When $p = 1$ and

$$\varphi(z) = \frac{1+Az}{1+Bz} \quad (-1 \leqslant B \leqslant A \leqslant 1),$$

inequality (15) reduces to give the inequality [3, Theorem 4, p. 894].

For the class $R_{b,p,g}(\varphi)$, we have the following result.

Theorem 4. *Let $\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$. If $f(z)$ given by (1) belongs to $R_{b,p}(\varphi)$, then for any complex number μ,*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant |\gamma| \max\{1; |v|\},$$

where

$$v := \frac{\mu g_{p+1}^2}{g_{p+2}} \frac{pbB_1(p+2)}{(p+1)^2} - \frac{B_2}{B_1} \quad \text{and} \quad \gamma := \frac{bpB_1}{g_{p+2}(2+p)}.$$

Further,

$$|a_{p+3}| \leqslant \frac{|b|pB_1}{(3+p)g_{p+3}} H(q_1, q_2), \tag{18}$$

where $H(q_1, q_2)$ is as defined in Lemma 3,

$$q_1 := \frac{2B_2}{B_1} \quad \text{and} \quad q_2 := \frac{B_3}{B_1}.$$

These results are sharp.

We now prove the coefficient bounds for $S^*_{p,g}(\alpha,\varphi)$.

Theorem 5. *Let* $\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$ *Further let*

$$\sigma_1 := \frac{(1+\alpha(p+1))[(1+\alpha(p+1))(B_2-B_1)+pB_1^2]}{2pB_1^2},$$

$$\sigma_2 := \frac{(1+\alpha(p+1))[(1+\alpha(p+1))(B_2+B_1)+pB_1^2]}{2pB_1^2},$$

$$\sigma_3 := \frac{(1+\alpha(p+1))[(1+\alpha(p+1))B_2+pB_1^2]}{2pB_1^2},$$

$$v := \frac{2\mu pB_1(p\alpha+2\alpha+1)}{(1+\alpha(p+1))^2} - \frac{pB_1}{1+\alpha(p+1)} - \frac{B_2}{B_1}, \qquad \gamma := \frac{pB_1}{2(1+\alpha(p+2))}.$$

If $f(z)$ *given by* (1) *belongs to* $S^*_p(\alpha,\varphi)$, *then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leqslant \begin{cases} -\gamma v & \text{if } \mu \leqslant \sigma_1, \\ \gamma & \text{if } \sigma_1 \leqslant \mu \leqslant \sigma_2, \\ \gamma v & \text{if } \mu \geqslant \sigma_2. \end{cases} \tag{19}$$

Further, if $\sigma_1 \leqslant \mu \leqslant \sigma_3$, *then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{(1+\alpha(p+1))^2\gamma}{p^2B_1^2}(v+1)|a_{p+1}|^2 \leqslant \gamma.$$

If $\sigma_3 \leqslant \mu \leqslant \sigma_2$, *then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{(1+\alpha(p+1))^2\gamma}{p^2B_1^2}(1-v)|a_{p+1}|^2 \leqslant \gamma.$$

For any complex number μ,

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leqslant \gamma \max\{1;|v|\}. \tag{20}$$

Further,

$$|a_{p+3}| \leqslant \frac{pB_1}{\alpha(5-p-p^2)+3}H(q_1,q_2), \tag{21}$$

where $H(q_1,q_2)$ *is as defined in Lemma* 3,

$$q_1 := \left[\frac{2B_2}{B_1} + \frac{B_1^2p(\alpha(3p+5)+3)}{2(\alpha(p+1)+1)(\alpha(p+2)+1)}\right]$$

and

$$q_2 := \left[\frac{B_3}{B_1} + \frac{B_2p(\alpha(3p+5)+3)+p^2B_1^2(p\alpha+\alpha+1)}{2(\alpha(p+1)+1)(\alpha(p+2)+1)}\right].$$

These results are sharp.

Proof. If $f(z) \in S^*_p(\alpha,\varphi)$. It is easily shown that

$$\begin{aligned}\frac{1+\alpha(1-p)}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\frac{z^2f''(z)}{f(z)} := {} & 1 + \frac{(1+\alpha(p+1))}{p}a_{p+1}z + \left(\frac{(1+\alpha(p+2))}{p}2a_{p+2} - \frac{(1+\alpha(p+1))}{p}a_{p+1}^2\right)z^2 \\ & + \left(\frac{(3+\alpha(5-p-p^2))}{p}a_{p+3} - \frac{(3+\alpha(3p+5))}{p}a_{p+1}a_{p+2} + \frac{(1+\alpha(p+1))}{p}a_{p+1}^3\right)z^3 \\ & + \cdots \end{aligned} \tag{22}$$

The proof can now be completed as in the proof of Theorem 1. □

For the class $S^*_{p,g}(\alpha,\varphi)$, we have the following result.

Theorem 6. *Let* $\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$, *and let*

$$\sigma_1 := \frac{g_{p+1}^2(1+\alpha(p+1))[(1+\alpha(p+1))(B_2-B_1)+pB_1^2]}{2pB_1^2 g_{p+2}},$$

$$\sigma_2 := \frac{g_{p+1}^2(1+\alpha(p+1))[(1+\alpha(p+1))(B_2+B_1)+pB_1^2]}{2pB_1^2 g_{p+2}},$$

$$\sigma_3 := \frac{g_{p+1}^2(1+\alpha(p+1))[(1+\alpha(p+1))B_2+pB_1^2]}{2pB_1^2 g_{p+2}},$$

$$v := \frac{2g_{p+2}\mu pB_1(p\alpha+2\alpha+1)}{g_{p+1}^2(1+\alpha(p+1))^2} - \frac{pB_1}{1+\alpha(p+1)} - \frac{B_2}{B_1}, \qquad \gamma := \frac{pB_1}{2g_{p+2}(1+\alpha(p+2))}.$$

If f(z) given by (1) *belongs to* $S^*_{p,g}(\alpha,\varphi)$, *then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leqslant \begin{cases} -\gamma v & \text{if } \mu \leqslant \sigma_1, \\ \gamma & \text{if } \sigma_1 \leqslant \mu \leqslant \sigma_2, \\ \gamma v & \text{if } \mu \geqslant \sigma_2. \end{cases}$$

Further, if $\sigma_1 \leqslant \mu \leqslant \sigma_3$, *then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{(1+\alpha(p+1))^2 g_{p+1}^2\gamma}{p^2B_1^2}(v+1)|a_{p+1}|^2 \leqslant \gamma.$$

If $\sigma_3 \leqslant \mu \leqslant \sigma_2$, *then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{(1+\alpha(p+1))^2 g_{p+1}^2\gamma}{p^2B_1^2}(1-v)|a_{p+1}|^2 \leqslant \gamma.$$

For any complex number μ,

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leqslant \gamma\max\{1;|v|\}.$$

Further,

$$|a_{p+3}| \leqslant \frac{pB_1}{g_{p+3}[\alpha(5-p-p^2)+3]}H(q_1,q_2), \tag{23}$$

where $H(q_1,q_2)$ *is as defined in* Lemma 3,

$$q_1 := \left[\frac{2B_2}{B_1} + \frac{B_1^2p(\alpha(3p+5)+3)}{2(\alpha(p+1)+1)(\alpha(p+2)+1)}\right]$$

and

$$q_2 := \left[\frac{B_3}{B_1} + \frac{B_2p(\alpha(3p+5)+3)+p^2B_1^2(p\alpha+\alpha+1)}{2(\alpha(p+1)+1)(\alpha(p+2)+1)}\right].$$

These results are sharp.

Remark 2. When $p = 1$ and

$$\varphi(z) = \frac{1+z(1-2\beta)}{1-z} \quad (\alpha \geqslant 0, 0 \leqslant \beta < 1),$$

(19) and (20) of Theorem 5 reduces to [4, Theorem 4 and 3, p. 95].

For the class $L_p^M(\alpha,\varphi)$, we now get the following coefficient bounds:

Theorem 7. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *Let*

$$\sigma_1 := \frac{\gamma_3(B_2 - B_1) + \gamma_2 B_1^2}{\gamma_1 B_1^2}, \qquad \sigma_2 := \frac{\gamma_3(B_2 + B_1) + \gamma_2 B_1^2}{\gamma_1 B_1^2}, \qquad \sigma_3 := \frac{\gamma_3 B_2 + \gamma_2 B_1^2}{\gamma_1 B_1^2},$$

$$\gamma_1 := 4p^2(p + 2 - 2\alpha), \qquad \gamma_2 := (1-\alpha)[2p(p+1)^2 + \alpha] + 2\alpha p^3, \qquad \gamma_3 := 2(p + 1 - \alpha)^2,$$

$$v := \frac{\gamma_1 \mu - \gamma_2}{\gamma_3} - \frac{B_2}{B_1}, \qquad T_1 := -\frac{3(p^2 + (1-\alpha)(3p+2))}{p^3},$$

$$T_2 := \frac{1-\alpha}{6p^6}[(p+1)^3[6p(p+\alpha) + \alpha(\alpha+1)] - p\alpha[-6p^3 + p^2(1+7\alpha) + 3p(1+3\alpha) + 3\alpha]] + \frac{\alpha}{p}.$$

If $f(z)$ *given by* (1) *belongs to* $L_p^M(\alpha, \varphi)$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant \begin{cases} \frac{-p^2 B_1 v}{2(p+2-2\alpha)} & \text{if } \mu \leqslant \sigma_1, \\ \frac{p^2 B_1}{2(p+2-2\alpha)} & \text{if } \sigma_1 \leqslant \mu \leqslant \sigma_2, \\ \frac{p^2 B_1 v}{2(p+2-2\alpha)} & \text{if } \mu \geqslant \sigma_2. \end{cases} \tag{24}$$

Further, if $\sigma_1 \leqslant \mu \leqslant \sigma_3$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{\gamma_3}{B_1 \gamma_1}(1 + v)|a_{p+1}|^2 \leqslant \frac{p^2 B_1}{2(p + 2 - 2\alpha)}. \tag{25}$$

If $\sigma_3 \leqslant \mu \leqslant \sigma_2$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{\gamma_3}{B_1 \gamma_1}(1 - v)|a_{p+1}|^2 \leqslant \frac{p^2 B_1}{2(p + 2 - 2\alpha)}. \tag{26}$$

For any complex number μ,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant \frac{p^2 B_1}{2(p + 2 - 2\alpha)} \max\{1; |v|\}. \tag{27}$$

Further,

$$|a_{p+3}| \leqslant \frac{p^2 B_1}{3(p - 3\alpha + 3)} H(q_1, q_2),$$

where $H(q_1, q_2)$ *is as defined in Lemma* 3,

$$q_1 := \frac{2B_2}{B_1} - \frac{T_1 p^4 B_1}{2(p - \alpha + 1)(p - 2\alpha + 2)}$$

and

$$q_2 := \frac{B_3}{B_1} - \frac{T_1 p^4(\gamma_3 B_2 + \gamma_2 B_1^2)}{2\gamma_3(p - \alpha + 1)(p - 2\alpha + 2)} - \frac{T_2 p^6 B_1^2}{(p - \alpha + 1)^3}.$$

These results are sharp.

Proof. The proof is similar to the proof of Theorem 1. □

Remark 3. As special cases, we note that for $p = 1$, inequalities (24)–(27) in Theorem 7 are those found in [15, Theorems 2.1 and 2.2, p. 3]. Additionally, if $\alpha = 1$, then inequalities (24)–(26) are the results established in [14, Theorem 2.1, Remark 2.2, p. 3].

Our final result is on the coefficient bounds for $M_p(\alpha, \varphi)$.

Theorem 8. *Let* $\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$ *Let*

$$\sigma_1 := \frac{\gamma_1(B_2 - B_1) + \gamma_2 B_1^2}{\gamma_3 B_1^2}, \qquad \sigma_2 := \frac{\gamma_1(B_2 + B_1) + \gamma_2 B_1^2}{\gamma_3 B_1^2}, \qquad \sigma_3 := \frac{\gamma_1 B_2 + \gamma_2 B_1^2}{\gamma_3 B_1^2},$$
$$\gamma_1 := p(1+p\alpha)^2, \qquad \gamma_2 := \alpha + p(p+2\alpha), \qquad \gamma_3 := 2p(p+2\alpha),$$
$$v := \frac{[p(p+2\alpha)(2\mu-1) - \alpha]B_1}{\gamma_3} - \frac{B_2}{B_1}.$$

If $f(z)$ *given by* (1) *belongs to* $M_p(\alpha,\varphi)$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant \begin{cases} \frac{-p^2B_1v}{2(p+2\alpha)} & \text{if } \mu \leqslant \sigma_1, \\ \frac{p^2B_1}{2(p+2\alpha)} & \text{if } \sigma_1 \leqslant \mu \leqslant \sigma_2, \\ \frac{p^2B_1v}{2(p+2\alpha)} & \text{if } \mu \geqslant \sigma_2. \end{cases} \tag{28}$$

Further, if $\sigma_1 \leqslant \mu \leqslant \sigma_3$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{\gamma_1}{B_1\gamma_3}(1+v)|a_{p+1}|^2 \leqslant \frac{p^2B_1}{2(p+2\alpha)}. \tag{29}$$

If $\sigma_3 \leqslant \mu \leqslant \sigma_2$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{\gamma_1}{B_1\gamma_3}(1-v)|a_{p+1}|^2 \leqslant \frac{p^2B_1}{2(p+2\alpha)}. \tag{30}$$

For any complex number μ,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant \frac{p^2B_1}{2(p+2\alpha)} \max\{1; |v|\}. \tag{31}$$

Further,

$$|a_{p+3}| \leqslant \frac{p^2B_1}{3(p+3\alpha)} H(q_1, q_2),$$

where $H(q_1,q_2)$ *is as defined in Lemma* 3,

$$q_1 := \frac{2B_2}{B_1} + \frac{3(p^2+3p\alpha+2\alpha)}{2(1+p\alpha)(p+2\alpha)} B_1,$$
$$q_2 := \frac{B_3}{B_1} + \frac{3(p^2+3p\alpha+2\alpha)}{2(1+p\alpha)(p+2\alpha)} B_2 + \frac{(p^4 + 5\alpha p^3 + 3p^2\alpha(2\alpha+1) + p\alpha(9\alpha-2) + 2\alpha^2)}{2p(1+p\alpha)^3(p+2\alpha)} B_1^2.$$

These results are sharp.

Proof. For $f(z) \in M_p(\alpha,\varphi)$, a computation shows that

$$\frac{1-\alpha}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + \frac{(1+p\alpha)a_{p+1}}{p} z + \left(-\frac{(p^2+2p\alpha+\alpha)a_{p+1}^2}{p^3} + \frac{2(p+2\alpha)a_{p+2}}{p^2}\right) z^2$$
$$+ \left(\frac{3(p+3\alpha)}{p^2} a_{p+3} - \frac{3(p^2+3p\alpha+2\alpha)}{p^3} a_{p+1}a_{p+2} + \frac{p^3+3p^2\alpha+3p\alpha+\alpha}{p^4} a_{p+1}^3\right) z^3 + \cdots$$

The remaining part of the proof is similar to the proof of Theorem 1. □

Remark 4. When $p = 1$ and $\varphi(z) = (\frac{1+z}{1-z})^{\beta}$ $(\alpha \geqslant 0, 0 < \beta \leqslant 1)$, (28) and (31) of Theorem 8 reduce to [2, Theorems 2.1 and 2.2, p. 23]. When $p = 1$ and $\alpha = 1$, (28)–(30) of Theorem 8 reduce to [9, Theorem 3 and Remark, p. 7].

Acknowledgement

The authors gratefully acknowledged support from the research grant IRPA 09-02-05-00020 EAR.

References

[1] S. Abdul Halim, On a class of functions of complex order, Tamkang J. Math. 30 (2) (1999) 147–153.
[2] N.E. Cho, S. Owa, On the Fekete-Szegö problem for strongly α-quasiconvex functions, Tamkang J. Math. 34 (1) (2003) 21–28.
[3] K.K. Dixit, S.K. Pal, On a class of univalent functions related to complex order, Indian J. Pure Appl. Math. 26 (9) (1995) 889–896.
[4] A. Janteng, M. Darus, Coefficient problems for certain classes of analytic functions, J. Inst. Math. Comput. Sci. Math. Ser. 13 (1) (2000) 91–96.
[5] F.R. Keogh, E.P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. 20 (1969) 8–12.
[6] K. Kiepiela, M. Pietrzyk, J. Szynal, The sharp bound for some coefficient functional within the class of holomorphic bounded functions and its applications, Rocky Mountain J. Math. 31 (1) (2001) 313–326.
[7] O.S. Kwon, N.E. Cho, On the Fekete-Szegö problem for certain analytic functions, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. 10 (4) (2003) 265–271.
[8] J.-L. Li, S. Owa, Sufficient conditions for starlikeness, Indian J. Pure Appl. Math. 33 (3) (2002) 313–318.
[9] W.C. Ma, D. Minda, A unified treatment of some special classes of univalent functions, in: Proceedings of the Conference on Complex Analysis (Tianjin, 1992), Internat. Press, Cambridge, MA, pp. 157–169.
[10] K.S. Padmanabhan, On sufficient conditions for starlikeness, Indian J. Pure Appl. Math. 32 (4) (2001) 543–550.
[11] D.V. Prokhorov, J. Szynal, Inverse coefficients for (α, β)-convex functions, Ann. Univ. Mariae Curie-Skłodowska Sec. A 35 (1981) 125–143, 1984.
[12] C. Ramesha, S. Kumar, K.S. Padmanabhan, A sufficient condition for starlikeness, Chinese J. Math. 23 (2) (1995) 167–171.
[13] V. Ravichandran, Y. Polatoglu, M. Bolcal, A. Sen, Certain subclasses of starlike and convex functions of complex order, Hacettepe J. Math. Stat. 34 (2005) 9–15.
[14] V. Ravichandran, A. Gangadharan, M. Darus, Fekete-Szegö inequality for certain class of Bazilevic functions, Far East J. Math. Sci. 15 (2) (2004) 171–180.
[15] V. Ravichandran, M. Darus, M. Hussain Khan, K.G. Subramanian, Differential subordination associated with linear operators defined for multivalent functions, Acta Math. Vietnam. 30 (2) (2005) 113–121.
[16] V. Ravichandran, Certain applications of first order differential subordination, Far East J. Math. Sci. 12 (1) (2004) 41–51.
[17] V. Ravichandran, C. Selvaraj, R. Rajalaksmi, Sufficient conditions for starlike functions of order α, J. Inequal. Pure Appl. Math. 3 (5) (2002). Article 81, 6 pp. (electronic).
[18] D.G. Yang, The Fekete-Szegö problem for Bazilevič functions of type α and order β, J. Math. Res. Expos. 18 (1) (1998) 99–104.